MATH 312
Lecture 24

THE PERRON - FROBENIUS THEOREM.

Recap:

- Stochastic Matrices < ≥ 0 entries / columns add to 1.
- Markov Chains: A stochastic, p_0's entries add to 1, then $(p_0, p_1, \ldots)$ is $p_k = A^k p_0$.
- Steady-State $p_\infty = \lim_{k \to \infty} p_k$ —
  - May NOT exist: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
  - May NOT be unique: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Thm If A is stochastic and has an eigenvalue $\lambda_1 = 1$ with all other $|\lambda| < 1$, then EVERY Markov chain of A has the same steady-state: this steady state is an eigenvector of A corresponding to $\lambda_1 = 1$.

TODAY What does it TAKE for a matrix to satisfy these conditions?

NOTATION: Just for this lecture, we write

- $\vec{v} \geq \vec{w}$ among vectors (or $A \geq B$ among matrices) to indicate "each entry v_i is $\geq$ the corresponding entry w_i", and similarly "each $A_{ij} \geq$ the corresp. B_{ij}". So, for instance:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \geq \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}.$$

But No relation between $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

- By $|\vec{v}|$ we mean NOT the length, but the vector of absolute values $\begin{bmatrix} |v_1| \\ \vdots \\ |v_n| \end{bmatrix}$. If $A = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$, then $|A| = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$, etc.

Def The $\boxed{\text{SPECTRAL RADIUS}}$ $\rho(A)$ of an $n\times n$ matrix A is defined as

$$\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$$

If $\rho(A)=0$ then ALL of A's eigenvalues are ZERO

Eg: $\rho\begin{bmatrix}0 & 1\\ 2 & -3\end{bmatrix} = \max\{0, |3|\} = 3$,

$\rho\begin{bmatrix}3+2i & 0\\ 0 & 5+6i\end{bmatrix} = |5+6i| = \sqrt{25+36} = \sqrt{61}$

$\rho\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} = 0$; $\rho(\tfrac{1}{n}A) = \tfrac{1}{n}\rho(A)$

Note: If A is diagonalizable and $\rho(A)=0$ then $A=0$. Just use $A=SDS^{-1}$ with $D=0$!

Prop 1 (Positive eigenstuff for positive A)

If $A>0$ (strictly), then
- $\rho(A)$ is an eigenvalue of A, $\rho(A)>0$.
- If v is an eigenvector of A corresponding to $\rho(A)$, then $|v|>0$ is also an eigenvector!

Pf May as well assume $\rho(A)=1$ (otherwise, rescale A)
Now, if
$$Av = \lambda v \quad [\text{with } v\neq 0 \text{ and } |\lambda|=1]$$
then: $|Av| = |\lambda v| = |\lambda||v| = |v|$
But: $|Av| \leq |A||v| = A|v|$

So: $\boxed{|v| \leq A|v|}$

WANT to show that this is an EQUALITY. Then, we'd get $A|v| = 1\cdot|v|$, and the $|v|$ has to be >0 because we assumed $A>0$!!

So assume (for contradiction) that $|v| < A|v|$ strictly. Then, $u = A|v| - |v|$ is >0. So, both $Au>0$ and $A|v|>0$. Then, (this is *slightly* tricky): There must be some tiny $d>0$ so that

$$Au > dA|v| > 0$$

$$\Rightarrow \quad A(A|v|-|v|) > dA|v| > 0$$

$$\Rightarrow \quad A^2|v| > (1+d)A|v| > 0$$

$$\Rightarrow \quad \frac{A^2|v|}{(1+d)} > A|v| > 0$$

Given $\vec{a}>0$ and $\vec{b}>0$, $\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

Pick $d_i < a_i/b_i$ for all i

Then set $d = \min_i (d_i)$

By construction,

$\therefore \quad \vec{a} > b\,\vec{d}$

Set $w = A|v|$ and $B = \frac{1}{1+d}A$. Then, this becomes

$$Bw > w > 0$$

So, $\quad w < Bw < B^2w < \cdots < B^kw < \cdots$

<u>BUT</u> $\quad \lim_{k\to\infty} B^k w = 0, \quad$ since $\rho(B^k) = \left(\frac{1}{1+d}\right)^k \to 0$ as $k\to\infty$.

So $w<0$, a contradiction!

Prop 2. If A is $n\times n$ and $A>0$, then for each eigenvalue λ of A,

$$|\lambda| = \rho(A) \implies \lambda = \rho(A)$$

i.e., no eigenvalues of the form $-\rho(A)$, or complex numbers with $|\lambda| = \rho(A)$, etc.

<u>Pf</u> Again, assume $\rho(A) = 1$. By Prop 1, if there is an eigenvalue λ of A with $|\lambda| = 1$, then we can select an ~~eigenvector~~ $v\neq 0$ so that

$$0 < |v| = A|v|.$$

$A = [A_{ij}]$

The first component v_1 of v satisfies

$$0 < |v_1| = (A|v|)_1 = \sum_{j=1}^{n} A_{1j}|v_j| \rightarrow (\alpha)$$

But since $|\lambda v| = |v|$, we also get

$$|v_1| = |\lambda v_1| = |(Av)_1| = \left|\sum_{j=1}^{n} A_{1j} v_j\right| \longrightarrow (\beta)$$

<u>FACT:</u> If $x_1, x_2, \ldots, x_n$ are nonzero complex numbers with $|x_1 + \cdots + x_n| = |x_1| + \cdots + |x_n|$, then EACH x_j is a positive multiple of x_1

By $(\alpha) = (\beta)$, we get

$$A_{11}|v_1| + \cdots + A_{1n}|v_n| = |A_{11}v_1 + \cdots + A_{1n}v_n|$$

By the "FACT" above, there are positive numbers $\pi_2, \pi_3, \ldots, \pi_n > 0$ so that

$$A_{1j}v_j = \pi_j (A_{11}v_1) \quad \text{for } j \geq 2$$

$$\Rightarrow \quad v_j = \underbrace{\left[\pi_j \cdot \frac{A_{11}}{A_{1j}}\right]}_{\text{This is } > 0\,!} v_1$$

Meaning

$$\vec{v} = \underbrace{\begin{bmatrix} 1 \\ \pi_2 A_{11}/A_{12} \\ \vdots \\ \pi_n A_{11}/A_{1n} \end{bmatrix}}_{\text{call this } \vec{u} > 0} v_1$$

$\boxed{\text{scalar mult. of } \vec{v}\,!!}$

Now, since $|\lambda| = 1$, we get

$$\lambda\vec{u} = A\vec{u} = |A\vec{u}| = |\lambda\vec{u}| = |\lambda|\vec{u} = \vec{u}$$

$\boxed{\text{because } \vec{u} > 0}$

Since $\lambda\vec{u} = \vec{u}$ and the first component of $\vec{u}$ is 1, we get $\lambda = 1$ ... Done!!

Prop 3 (Multiplicity 1) If $A > 0$ is $n \times n$, then $\dim N(A - \rho(A) I) = 1$, so $\rho(A)$ has at most <u>ONE</u> linearly independent eigenvector.

eigenvecs of A with e-val $\rho(A)$

Pf If not, we have independent $\vec{x}, \vec{y}$ that may be chosen > 0 by Prop 1. Set $\vec{z} = (y_1/x_1)\vec{x}$ and note $\vec{z} \neq \vec{y}$ by the assumption of linear independence. Now,

So, $z_1 = y_1$ But $z \neq y$

$$A(z - y) = \rho(A)(z - y)$$

But the first entry of both vectors is zero, so $\rho(A)$ must be zero: this violates Prop 1.

QUICK NOTE

$\rho(A^T) = \rho(A)$: in fact, the EIGENVALUES of A and A^T always agree, because

$$\det(A - \lambda I) = 0 \iff \det(A^T - \lambda I) = 0$$
$$\iff \det(A - \lambda I)^T = 0$$

Prop 4 If $A \geq 0$ is STOCHASTIC, then $\rho(A) = 1$.

Pf First, note $\rho(A) \leq 1$: If λ is an eigenvalue, then $Av = \lambda v$ for $v \neq 0$. Rescale v so that its entries sum to 1, and then note Av's entries also sum to 1. So, λv's entries must sum to 1, which forces $|\lambda| \leq 1$.

Next, to see $\rho(A) = 1$, we see that 1 is <u>always</u> an eigenvalue if A is stochastic! For this, use the QUICK NOTE above and the fact that

$$A^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \text{Sum of row 1 of } A^T \\ \vdots \\ \text{Sum of row n of } A^T \end{bmatrix}$$

But since rows of A^T = cols of A sum to 1,

$$A^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

So $\lambda = 1$ is an eigenvalue of A^T with eigenvector $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, which means 1 is ALSO an eigenvalue of A by the QUICK NOTE. So, $\rho(A) = 1$

<u>Prop 5</u> If $A > 0$ is $n \times n$, then no eigenvalue of A except for $\rho(A)$ can have an eigenvector with STRICTLY positive entries.

<u>Pf</u> Assume $\lambda \neq \rho(A)$ is an eigenvalue of A with eigenvector $y > 0$. Then,

$$Ay = \lambda y \qquad \rightarrow (1)$$

By Prop 1, $\rho(A)$ is an eigenvalue of A and by QUICK NOTE, it is also an eigenvalue of A^T. Let $w > 0$ be the eigenvector of A^T corresponding to $\rho(A)$, so:

$$A^T w = \rho(A) w \qquad \rightarrow (2)$$

Multiply (1) by w^T to get

$$w^T A y = w^T \lambda y$$

And transpose:

$$y^T A^T w = y^T \lambda w$$

By (2), $A^T w = \rho(A) w$ so

$$y^T \rho(A) w = y^T \lambda w.$$

$$\Rightarrow \rho(A)\left(y^T w\right) = \lambda \left(y^T w\right)$$

Now, $y > 0$ and $w > 0$ so $y^T w > 0$ and it can be cancelled, leaving $\underline{\lambda = \rho(A)}$. This contradicts $\lambda \neq \rho(A)$ from the first line of this proof.

PUTTING IT ALL TOGETHER: Props 1, 2, 3, 5.

Thm [Perron-Frobenius] Let $A > 0$ be $n \times n$.
There are $\underline{4}$ consequences of the strict positivity of A:

1. $\rho(A)$ is a non-repeated eigenvalue of A.
2. All other eigenvalues satisfy $|\lambda| < \rho(A)$ strictly.
3. $\rho(A)$ has a STRICTLY POSITIVE eigenvector $\vec{w} > 0$ (at most one linearly independent choice)
4. No other eigenvalue of A has a strictly positive eigenvector

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<u>Note</u> If we scale $\vec{w} > 0$ (the positive eigenvector) so that its entries add to 1 then it is called the $\boxed{\text{PERRON EIGENVECTOR}}$ of A.

Application to STOCHASTIC MATRICES / Markov Chains

- If A is stochastic AND strictly > 0, then all its Markov chains have the Perron eigenvector as the steady state!